Single-point velocity distribution in turbulence

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Abstract

We show that the tails of the single-point velocity probability distribution function (PDF) are generally non-Gaussian in developed turbulence. By using instanton formalism for the Navier-Stokes equation, we establish the relation between the PDF tails of the velocity and those of the external forcing. In particular, we show that a Gaussian random force having correlation scale L and correlation time τ produces velocity PDF tails $\ln\mathcal{P}(v) \propto -v^4$ at $v \gg v_{\rm rms}, L/\tau$. For a short-correlated forcing when $\tau \ll L/v_{\rm rms}$ there is an intermediate asymptotics $\ln\mathcal{P}(v) \propto -v^3$ at $L/\tau \gg v \gg v_{\rm rms}$.

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Early experimental data on skewness and flatness of the velocity field prompted one to believe that the single-point velocity PDF in developed turbulence is generally close to Gaussian [1,2]. A possible reasoning may be that large-scale motions (that give the main contribution into velocity statistics at a point) are connected to a random external forcing f then velocity $v(t) = \int_0^t f(t')dt'$ has to be Gaussian if t is larger then the correlation time τ of the forcing, irrespective of the statistics of f. That would be the case if the force was the only agent affecting velocity. However, there are also nonlinearity (leading to instability and break-up of large-scale motions) and viscosity (that dissipates small-scale modes appeared as a result of the instability). Let us first explain the simple physics involved and formulate the predictions following from physical arguments, then we develop the formalism which gives the predicted PDF tails. Qualitatively, one may describe the interplay between external force and nonlinearity in the following way. Force f pumps velocity $v \sim ft$ until the time $t_* \sim L/v$ when nonlinearity restricts the growth. The relation between velocity and forcing can thus be suggested as follows: $v^2 \sim fL$. Therefore, velocity's PDF can be obtained by substituting $f \sim v^2/L$ into force's PDF $\mathcal{P}_f$: $\mathcal{P}(v) \sim \mathcal{P}_f(v^2/L)$. Those arguments presume that t_* is less than the correlation time τ of forcing. If opposite is the case $t_* \gg \tau$ then the law of velocity growth is different $v^2 \sim f^2 t\tau$, so that the velocity increases up to $v^3 \sim f^2 L\tau$; the short-correlated pumping is effectively Gaussian $\mathcal{P}_f \sim \exp(-f^2)$ and the velocity's PDF is $\mathcal{P}(v) \sim \exp[-(v/v_{\rm rms})^3]$. We see that velocity PDF is expected to be dependent on the statistics of the force.

Actual mechanism of restriction for the Navier-Stokes equation (instability of a large-scale flow leading to a cascade that provides for a viscous dissipation) is irrelevant for the above arguments. What matters is that we deal with the system of the hydrodynamic type so that nonlinear time t_* can be estimated as L/v. For example, the same argument goes for Burgers equation where t_* is a breaking time and viscous dissipation at a shock prevents further growth [5]. It is interesting that viscosity does not enter above estimates yet it's existence is crucial for the whole picture to be valid.

Let us stress that the above arguments can be only applied to rare events with velocity

and force being much larger than their root-mean-square values when the influence of background fluctuations can be neglected. The above predictions are thus made for PDF tails. For a a nonlinear dissipative system, it is generally difficult to relate an output statistics to the statistics of the input (be it initial conditions or external force). Our aim here is to show that it is possible, nevertheless, to relate the probabilities of rare fluctuations that is to relate the tails of the PDFs of the force and the field that is forced respectively. A rigorous way to describe rare fluctuations is the instanton formalism recently developed for turbulence [3] and employed for obtaining PDF tails in different problems [4–6].

The main idea of the method is that the tails are described by saddle-point configurations in the path integral for the correlation functions of the turbulent variable (say, velocity $\boldsymbol{v}$). We call the configuration instanton because of a finite lifetime. One may call it also optimal fluctuation since it corresponds to the extremum of the probability.

We start with the Navier-Stokes equation

$$\partial_t v_\alpha + v_\beta \nabla_\beta v_\alpha - \nu \nabla^2 v_\alpha + \nabla_\alpha P = f_\alpha \,, \tag{1}$$

where $\boldsymbol{f}$ is a random force (per unit mass) pumping the energy into the system and ν is the viscosity coefficient. Incompressibility is assumed so that $\operatorname{div} \boldsymbol{v} = \operatorname{div} \boldsymbol{f} = 0$. The field P in (1) is the pressure divided by the mass density ρ. Velocity correlation functions can be presented as path integrals which form is determined by the statistics of pumping. Let us first consider a Gaussian forcing with the correlation function $\langle f_\alpha(t, \boldsymbol{r}) f_\beta(t', \boldsymbol{r}') \rangle = \Xi_{\alpha\beta}(t-t', \boldsymbol{r}-\boldsymbol{r}')$ which is assumed to decay on the scale τ as a function of the first argument and on the scale L as a function of the second one. Then moments of the velocity can be written as path integrals:

$$\langle v^{2n} \rangle = \int \mathcal{D}p\, \mathcal{D}v\, \mathcal{D}P\, \mathcal{D}Q \, \exp\left(i\mathcal{I} + 2n \ln v\right) \,, \tag{2}$$

where $\boldsymbol{p}$ is an auxiliary field and the effective action has the following form [7].

$$\begin{aligned} \mathcal{I} = \int dt\, d\boldsymbol{r}\, [&p_\alpha(\partial_t v_\alpha + v_\beta \nabla_\beta v_\alpha - \nu \nabla^2 v_\alpha + \nabla_\alpha P) \\ &+ Q \nabla_\alpha v_\alpha] + \frac{i}{2} \int dt' dt d\boldsymbol{r}' d\boldsymbol{r}\, \Xi_{\alpha\beta}(t-t', \boldsymbol{r}-\boldsymbol{r}') p_\alpha p'_\beta \,. \end{aligned} \tag{3}$$

The independent fields P and Q play the role of Lagrange multipliers enforcing the incompressibility conditions on the velocity and the response field: $\nabla_\alpha v_\alpha = \nabla_\alpha p_\alpha = 0$.

The tails of the velocity PDF are determined by high moments with $n \gg 1$ which can be found by applying the saddle-point method to the integral (2): $\langle v^{2n} \rangle = u^n(0,0)\exp(\mathcal{I}_{extr})$. The configuration $\boldsymbol{u}(t,\boldsymbol{r})$, $\boldsymbol{p}(t,\boldsymbol{r})$, $P(t,\boldsymbol{r})$ and $Q(t,\boldsymbol{r})$ corresponding to a saddle point is our instanton. The extremum conditions for the argument of the exponent in (2) determining the instanton give two dynamical equations

$$\begin{aligned} &\partial_t u_\alpha + u_\beta \nabla_\beta u_\alpha - \nu \nabla^2 u_\alpha + \nabla_\alpha P \\ &= -i \int dt'\, d\boldsymbol{r'}\, \Xi_{\alpha\beta}(t-t', \boldsymbol{r}-\boldsymbol{r'}) p_\beta(t', \boldsymbol{r'})\,, \end{aligned} \tag{4}$$

$$\begin{aligned} &\partial_t p_\alpha - p_\beta \nabla_\alpha u_\beta + u_\beta \nabla_\beta p_\alpha + \nu \nabla^2 p_\alpha + \nabla_\alpha Q \\ &= 2\, i\, n\, \delta(t)\delta(\boldsymbol{r}) u_\alpha / u^2\,, \end{aligned} \tag{5}$$

and two incompressibility constraints

$$\nabla^2 P = -\nabla_\alpha (u_\beta \nabla_\beta u_\alpha)\,. \tag{6}$$

$$\nabla^2 Q = \nabla_\alpha (p_\beta \nabla_\alpha u_\beta - u_\beta \nabla_\beta p_\alpha)\ . \tag{7}$$

As it was explained in [3], the system (4–7) is to be solved at negative time because it describes the prehistory that leads to a given measured value of the velocity at $t=0$. Since $p=0$ at $t>0$ then the right-hand side of (5) directly produces the value of p at $t=-0$:

$$p_\alpha(0,\boldsymbol{r}) = -2in\left[\delta_{\alpha\beta}\delta(\boldsymbol{r}) - \frac{1}{4\pi r^3}\left(\delta_{\alpha\beta} - \frac{3r_\alpha r_\beta}{r^2}\right)\right]\frac{u_\beta(0,0)}{u^2(0,0)}\ . \tag{8}$$

The equations (4,5) are to be solved at $-\infty < t < 0$ with the constraints (6,7) under the boundary conditions (8) and $u \to 0$ as $t \to -\infty$. Instanton approach thus reduces the statistical problem (finding PDF tails) to the dynamical problem of finding a particular solution of the deterministic equations (4,5). Note that the instanton equations are deterministic they present already substantial simplification with respect to the initial Navier-Stokes equation with a random force (which is not suprising because they describe only part of velocity

statistics, namely the tails of the single-point PDF). Still, the system (4,5) is quite complicated and it's complete solution (which will provide an important information about the spatio-temporal domains with high velocity) is still ahead of us. The main difficulty is an effective spatial nonlocality of the constraints (6,7).

Our goal in this paper is to show that a definite statement about the functional form of velocity PDF tails can be obtained from the analysis of the symmetries of the instanton equations without actually finding a solution. Let us show how the symmetry analysis gives n-dependence of $\langle v^{2n}\rangle$ which determines the functional form of PDF tails. What is important to state is that both fields $\boldsymbol{u}$ and $\boldsymbol{p}$ decay at moving backwards in time during some characteristic time t_* which we call the lifetime of the instanton. Actually it is the same time which we qualitatively discussed above at treating rare events with strong forcing since the instanton presents just the space-time picture of those typical events contributing to $\langle u^{2n}\rangle$. Symmetry analysis depends on whether the pumping correlation time τ is larger or smaller than the instanton lifetime t_*. Let us first consider the case $\tau \gg t_*$, that makes it possible to consider the pumping correlation function Ξ as time independent. In this case, the parameter n can be excluded from (4–7) by the rescaling transformation

$$\begin{aligned} &t \to X^{-1} t\,, \quad \boldsymbol{u} \to X\boldsymbol{u}\,, \quad P \to X^2 P\,, \quad Q \to X^4 Q\,, \\ &\nu \to X\nu\,, \quad \boldsymbol{p} \to X^3 \boldsymbol{p}\,, \quad X^4 = n\,. \end{aligned} \tag{9}$$

That gives a general n-dependence of the velocity

$$\boldsymbol{u} = n^{1/4} \boldsymbol{\varphi}(\nu^4/n)\,, \tag{10}$$

where dimensionless function φ is expected to go to some constant when it's argument goes to zero. This is equivalent to the physically plausible assumption that the high velocity moments are viscosity independent. Under such an assumption the n-dependence of the instanton solution $u \propto n^{1/4}$ gives the following n-dependence of the moment $\langle v^{2n}\rangle \propto n^{n/2}$ which corresponds to the PDF tail

$$\ln \mathcal{P}(v^2) \propto -v^4\,. \tag{11}$$

Note that the integral term in the action $\mathcal{I}_{extr} \propto n$ i.e. the factor $\exp(\mathcal{I}_{extr})$ gives only subleading contribution into (11). If $L/\tau \lesssim v_{\rm rms}$ (where $v_{\rm rms}$ is the typical value of the velocity fluctuations) then the asymptotics (11) is realized at $v \gg v_{\rm rms}$ and $\ln \mathcal{P} \sim -(v/v_{\rm rms})^4$. In the opposite limit of a short-correlated pumping with $L/\tau \gg v_{\rm rms}$, an intermediate asymptotics exists where the pumping correlation function Ξ can be treated as delta-correlated in time: $\Xi(t, \boldsymbol{r}) = \delta(t)\chi(\boldsymbol{r})$. Then the only changes in (9) are

$$\boldsymbol{p} \to X^2 \boldsymbol{p}\,, \quad Q \to X^3 Q\,, \quad X^3 = n\,. \tag{12}$$

That leads to the law (cf. [3])

$$\ln \mathcal{P}(v^2) \sim -(v/v_{\rm rms})^3\,, \tag{13}$$

which is valid at $L/\tau \gg v \gg v_{\rm rms}$. For larger v, the asymptotics (11) is realized. We thus see that for Gaussian pumping the velocity PDF always decreases faster than Gaussian.

Note that the above consideration can be readily extended for the consideration of velocity differences $w(\mathbf{r}) = |\boldsymbol{v}(\mathbf{r}) - \boldsymbol{v}(\mathbf{0})|$ at any r. There is an intermediate region $w_{rms}(r) \ll w(r) \ll v_{\rm rms}$ where $\mathcal{P}(w)$ is not described by the direct instanton formalism. The same formulas (11,13) as well as below (20) describe the remote tails of $\mathcal{P}(w)$ at $w \gg v_{\rm rms}$, which are probably unaccessible experimentally at present.

Contrary, faster-than-Gaussian decay of a single-point velocity PDF was recently observed both in experiments [8] and in numerical simulations [9–11], the qualitative reason for that (short life time of strong fluctuations) was discussed in [11]. Unfortunately, except the qualitative statement on faster-than Gaussian decay there were no definite data in [9–11] to allow for a quantitative comparison with (11,13), this remains for the future work.

The transformations (9,12) shows that the lifetime of the instanton is short for large n: $t_* \propto n^{-1/3}$ for the fast pumping $\tau \ll t_*$ and $t_* \propto n^{-1/4}$ for the slow pumping $\tau \gg t_*$. Therefore at large enough n we always deal with a slow pumping. In other words, distant tails of the velocity PDF are determined by a simultaneous statistics of the pumping (which is not necessary Gaussian) rather than by integral over time as one would naively expect.

As we shall show now, the same property ($\tau \gg t_*$) is correct for a non-Gaussian statistics of the pumping. Indeed, the above procedure can be readily generalized for an arbitrary pumping statistics when instead of (2) one has

$$\langle |\boldsymbol{v}|^{2n} \rangle = \int \mathcal{D}f \mathcal{D}p \mathcal{D}v \mathcal{D}P \mathcal{D}Q \mathcal{P}_f \exp\left(i\tilde{\mathcal{I}} + 2n \ln |\boldsymbol{v}|\right) \,. \tag{14}$$

Here, we substituted the simultaneous PDF of pumping $\mathcal{P}_f(f)$ since we treat the slow pumping and the effective action $\tilde{\mathcal{I}}$ figuring in (14) for the Navier-Stokes equation has now the following form

$$\begin{aligned} \tilde{\mathcal{I}} = \int dt\, d\boldsymbol{r}\, [p_\alpha(\partial_t v_\alpha + v_\beta \nabla_\beta v_\alpha - \nu \nabla^2 v_\alpha + \nabla_\alpha P) \\ + Q \nabla_\alpha v_\alpha - p_\alpha f_\alpha] \,. \end{aligned} \tag{15}$$

The saddle-point equation for the instanton velocity $\boldsymbol{u}$ is now written as

$$\partial_t u_\alpha + u_\beta \nabla_\beta u_\alpha - \nu \nabla^2 u_\alpha + \nabla_\alpha P = f_\alpha \,, \tag{16}$$

with the relation

$$\int dt\, p_\alpha = -i \delta \ln \mathcal{P}_f(f) / \delta f_\alpha \,. \tag{17}$$

The equation for $\boldsymbol{p}$ and the incompressibility constraints are the same (5,6,7).

Let us assume the tail of $\mathcal{M} = -\ln \mathcal{P}_f$ to be scale invariant with the exponent a:

$$\mathcal{M}(X\boldsymbol{f}) = X^a \mathcal{M}(\boldsymbol{f}) \,, \tag{18}$$

Now, we can generalize (9)

$$\begin{aligned} & t \to X^{-1} t \,, \quad \boldsymbol{u} \to X \boldsymbol{u} \,, \quad P \to X^2 P \,, \\ & \nu \to X \nu \,, \quad \boldsymbol{f} \to X^2 \boldsymbol{f} \,, \\ & \boldsymbol{p} \to X^{2a-1} \boldsymbol{p} \,, \quad Q \to X^{2a} Q \,, \quad X^{2a} = n \,, \end{aligned} \tag{19}$$

keeping the form of the equations (16), (17) and removing n from the equation (5). Using the law (19) we can find how the characteristic velocity scales with n. As above the answer

can be expressed in terms of the behavior of the tail of the simultaneous PDF $\mathcal{P}(v^2)$ for the velocity:

$$\ln \mathcal{P}(v^2) \propto -v^{2a}\,. \tag{20}$$

The lifetime t_* of our instanton scales as $t_* \propto n^{-1/(2a)}$ that is decreases with n for any positive a which justifies our consideration. For Gaussian statistics $a = 2$ and we reproduce (11).

Note that the velocity PDF decays always faster than that of the force; in particular, Gaussian velocity would correspond to the force PDF decaying exponentially.

The expression (3) implies homogeneous pumping. Meanwhile, the transformations (9,12,19) do not transform coordinates and can be generalized for spatially ingomogeneous pumping statistics. Therefore our conclusions are true also for a more physical case of an inhomogeneous pumping, in particular acting on the boundaries of the flow.

To conclude, let us describe the status of the results obtained: Under the assumptions that the solutions of the instanton equations exist and the probability of finding very high velocity is viscosity independent we found how that probability is related to the statistics of the force.

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